Supplemental Materials for Invariant Multiparameter Sensitivity to Characterize Dynamical Systems on Complex Networks

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Analysis S1: Linear Dynamical Model

Let x_i be the position of particle i and b_i be its velocity with no interaction. The dynamics are described by

$$\dot{x}_i = b_i + \sum_{j=1}^{N} W_{ij}\left(x_j - x_i\right), \tag{S-1}$$

where the connection weight matrix $\mathbf{W}$ is a symmetric matrix, and $W_{ij} \geq 0$. By using a moving coordinate system, we can assume

$$\sum_{i=1}^{N} b_i = 0 \tag{S-2}$$

without loss of generality. In this case, we can assume that the mean of x_i is 0. In a matrix expression, we have

$$\dot{\vec{x}} = \vec{b} + \mathbf{W}\vec{x} - \mathbf{W}^d\vec{x}, \tag{S-3}$$

where

$$\mathbf{W}^d = \begin{pmatrix} \sum_{j=1}^{N} W_{1j} & 0 & 0 & \cdots & 0 \\ 0 & \sum_{j=1}^{N} W_{2j} & 0 & \cdots & 0 \\ 0 & 0 & \sum_{j=1}^{N} W_{3j} & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{j=1}^{N} W_{Nj} \end{pmatrix}. \tag{S-4}$$

Because the Laplacian matrix $\mathbf{L}$ defined by

$$\mathbf{L} = \mathbf{W}^d - \mathbf{W} \tag{S-5}$$

is non-negative definite, the positions of the particles converge to fixed points. After the convergence, we have

$$\dot{\vec{x}} = \vec{b} - \mathbf{L}\vec{x} = 0. \tag{S-6}$$

We regard the variance V of $\vec{x}$ as the output of this system. When all particles constitute a connected graph, the rank of $\mathbf{L}$ is $N-1$ [1]. Note that

$$\mathbf{L}\vec{1} = \vec{0}, \tag{S-7}$$

where $\vec{1} = [1, 1, \ldots, 1]^T$ and $\vec{0} = [0, 0, \ldots, 0]^T$. $\mathbf{L}$ is a real symmetric matrix. Thus, we can expand $\mathbf{L}$ as

$$\mathbf{L} = \sum_{i=1}^{N-1} \lambda_i \vec{u}_i \vec{u}_i^T, \tag{S-8}$$

where $\vec{u}_i$ is an eigenvector of $\mathbf{L}$ and λ_i is the eigenvalue corresponding to $\vec{u}_i$. Notably, $\vec{1} \perp \vec{u}_i$. We can not obtain the inverse of $\mathbf{L}$ to solve Eq. (S-1) because $\mathbf{L}$ is not full rank. Thus, we define a full-rank real symmetric matrix $\tilde{\mathbf{L}}$ by using a non-zero constant β:

$$\tilde{\mathbf{L}} = \mathbf{L} + \beta \mathbf{1}, \tag{S-9}$$

where

$$\mathbf{1} = \vec{1}\vec{1}^T. \tag{S-10}$$

By using $\tilde{\mathbf{L}}$, we can obtain a solution of Eq. (S-6) by

$$\vec{x} = \tilde{\mathbf{L}}^{-1}\vec{b}. \tag{S-11}$$

By multiplying $\vec{1}$ from the left in Eq. (S-3), we see that this solution satisfies $\sum x_i = 0$. The output V can be derived as follows:

$$\begin{aligned} V = \frac{\vec{x}\cdot\vec{x}}{N} &= \frac{1}{N}\vec{b}^T(\tilde{\mathbf{L}}^{-1})^T\tilde{\mathbf{L}}^{-1}\vec{b} \\ &= \frac{1}{N}\vec{b}^T\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{L}}^{-1}\vec{b}, \end{aligned} \tag{S-12}$$

because $\tilde{\mathbf{L}}^{-1}$ is also a symmetric matrix. In the following equations, we regard each coupling strength W_{ij} as a parameter. Because we have introduced the parameter β, the parameter sensitivity for β should also be taken into account. However V is independent of β because

$$\begin{aligned} \frac{\partial V}{\partial \beta} &= -\frac{1}{N}\vec{b}^T\tilde{\mathbf{L}}^{-1}\left(\frac{\partial \tilde{\mathbf{L}}}{\partial \beta}\tilde{\mathbf{L}}^{-1} + \tilde{\mathbf{L}}^{-1}\frac{\partial \tilde{\mathbf{L}}}{\partial \beta}\right)\tilde{\mathbf{L}}^{-1}\vec{b} \\ &= -\frac{1}{N}\vec{b}^T\tilde{\mathbf{L}}^{-1}\left(\mathbf{1}\tilde{\mathbf{L}}^{-1} + \tilde{\mathbf{L}}^{-1}\mathbf{1}\right)\tilde{\mathbf{L}}^{-1}\vec{b} \\ &= -\frac{1}{N}\vec{b}^T\tilde{\mathbf{L}}^{-1}\left(\frac{2N}{\beta}\mathbf{1}\right)\tilde{\mathbf{L}}^{-1}\vec{b} \\ &= -\frac{1}{N}\vec{b}^T\tilde{\mathbf{L}}^{-1}\frac{2N^2}{\beta^2}\mathbf{1}\vec{b} \\ &= -\frac{1}{N}\vec{b}^T\tilde{\mathbf{L}}^{-1}\frac{2N^2}{\beta^2}\vec{0} \\ &= 0, \end{aligned} \tag{S-13}$$

where we used

$$\frac{\partial \tilde{\mathbf{L}}^{-1}}{\partial \beta} = -\tilde{\mathbf{L}}^{-1}\frac{\partial \tilde{\mathbf{L}}}{\partial \beta}\tilde{\mathbf{L}}^{-1}, \tag{S-14}$$

$$\frac{\partial \tilde{\mathbf{L}}}{\partial \beta} = \mathbf{1}, \tag{S-15}$$

$$\tilde{\mathbf{L}}^{-1} = \sum_{i=1}^{N-1}\frac{1}{\lambda_i}\vec{u}_i\vec{u}_i^T + \frac{1}{\beta}\mathbf{1}, \tag{S-16}$$

$$\mathbf{1}\tilde{\mathbf{L}}^{-1} = \frac{N}{\beta}\mathbf{1}. \tag{S-17}$$

The IMPS is derived as

$$
\begin{aligned}
\mathrm{IMPS} &= \sum_{\langle ij\rangle} \left| \frac{W_{ij}}{V} \frac{\partial V}{\partial W_{ij}} \right| + \left| \frac{\beta}{V} \frac{\partial V}{\partial \beta} \right| \\
&= \frac{1}{N} \sum_{\langle ij\rangle} \left| \frac{W_{ij}}{V} \frac{\partial \vec{b}^T \tilde{\mathbf{L}}^{-1} \tilde{\mathbf{L}}^{-1} \vec{b}}{\partial W_{ij}} \right| + \frac{1}{N} \left| -\frac{\beta}{V} \vec{b}^T \tilde{\mathbf{L}}^{-1} \left(\mathbf{1} \tilde{\mathbf{L}}^{-1} + \tilde{\mathbf{L}}^{-1} \mathbf{1} \right) \tilde{\mathbf{L}}^{-1} \vec{b} \right| \\
&= \frac{1}{N} \sum_{\langle ij\rangle} \left| -\frac{W_{ij}}{V} \vec{b}^T \tilde{\mathbf{L}}^{-1} \left(\frac{\partial \tilde{\mathbf{L}}}{\partial W_{ij}} \tilde{\mathbf{L}}^{-1} + \tilde{\mathbf{L}}^{-1} \frac{\partial \tilde{\mathbf{L}}}{\partial W_{ij}} \right) \tilde{\mathbf{L}}^{-1} \vec{b} \right| \\
&\quad + \frac{1}{N} \left| -\frac{\beta}{V} \vec{b}^T \tilde{\mathbf{L}}^{-1} \left(\mathbf{1} \tilde{\mathbf{L}}^{-1} + \tilde{\mathbf{L}}^{-1} \mathbf{1} \right) \tilde{\mathbf{L}}^{-1} \vec{b} \right|, \qquad \text{(S-18)}
\end{aligned}
$$

where $\langle\rangle$ is the summation over the pairs of (i, j) with $W_{ij} \neq 0$. Here we assume that SPSs have the same sign. By using

$$
\sum_{\langle ij\rangle} W_{ij} \frac{\partial \tilde{\mathbf{L}}}{\partial W_{ij}} = \mathbf{L}, \qquad \text{(S-19)}
$$

we obtain

$$
\begin{aligned}
\mathrm{IMPS} &= \frac{1}{N} \Bigg| -\frac{1}{V} \vec{b}^T \tilde{\mathbf{L}}^{-1} [\mathbf{L} \tilde{\mathbf{L}}^{-1} + \tilde{\mathbf{L}}^{-1} \mathbf{L}] \tilde{\mathbf{L}}^{-1} \vec{b} \\
&\qquad - \frac{\beta}{V} \vec{b}^T \tilde{\mathbf{L}}^{-1} \left(\mathbf{1} \tilde{\mathbf{L}}^{-1} + \tilde{\mathbf{L}}^{-1} \mathbf{1} \right) \tilde{\mathbf{L}}^{-1} \vec{b} \Bigg| \\
&= \frac{1}{N} \left| -\frac{1}{V} \vec{b}^T \tilde{\mathbf{L}}^{-1} \left[(\mathbf{L} + \beta \mathbf{1}) \tilde{\mathbf{L}}^{-1} - \tilde{\mathbf{L}}^{-1} (\mathbf{L} + \beta \mathbf{1}) \right] \tilde{\mathbf{L}}^{-1} \vec{b} \right| \\
&= \frac{1}{N} \left| -\frac{1}{V} \vec{b}^T \tilde{\mathbf{L}}^{-1} [\tilde{\mathbf{L}} \tilde{\mathbf{L}}^{-1} + \tilde{\mathbf{L}}^{-1} \tilde{\mathbf{L}}] \tilde{\mathbf{L}}^{-1} \vec{b} \right| \\
&= \frac{1}{N} \left| -\frac{2}{V} \vec{b}^T \tilde{\mathbf{L}}^{-1} \tilde{\mathbf{L}}^{-1} \vec{b} \right| \\
&= \left| -\frac{2V}{V} \right| \\
&= 2. \qquad \text{(S-20)}
\end{aligned}
$$

Analysis S2: Nonlinear Model

The dynamics of phase oscillators are expressed as

$$
\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i), \qquad \text{(S-21)}
$$

where ω_i is the natural frequency of oscillator i, $\mathbf{K}$ is the symmetric connection weight matrix and $K_{ij} \geq 0$. We use the circular variance V_c of the oscillators [2]

$$
V_c = 1 - r = 1 - \frac{1}{N} \sqrt{C^2 + S^2} \qquad \text{(S-22)}
$$

in the phase-locked state as the output, where r is the Kuramoto order parameter, $C = \sum_{i=1}^{N} \cos\theta_i$ and $S = \sum_{i=1}^{N} \sin\theta_i$. We can assume $\sum_{i=1}^{N} \omega_i = 0$ without loss of generality. In the phase-locked state, the right-hand side y'_i of Eq. (S-21) is 0; i.e.,

$$\vec{y'} = \vec{0}. \tag{S-23}$$

Here we derive the relationship between the connection weights and the phases under the condition that Eq. (S-23) is satisfied. For a small change $\Delta\vec{\theta}$ of $\vec{\theta}$, we denote the resulting small change in $\mathbf{K}$ by $\Delta\mathbf{K}$. We obtain

$$\begin{aligned} y'_i + \Delta y'_i &= \omega_i + \sum_{j=1}^{N} (K_{ij} + \Delta K_{ij}) \sin(\theta_j + \Delta\theta_j - \theta_i - \Delta\theta_i) \\ &\approx \omega_i + \sum_{j=1}^{N} (K_{ij} + \Delta K_{ij}) \left[\sin(\theta_j - \theta_i) + \cos(\theta_j - \theta_i)(\Delta\theta_j - \Delta\theta_i)\right]. \end{aligned} \tag{S-24}$$

Subtracting y'_i from both sides yields

$$\begin{aligned} \Delta y'_i &\approx \sum_{j=1}^{N} K_{ij} \cos(\theta_j - \theta_i)(\Delta\theta_j - \Delta\theta_i) \\ &\quad + \sum_{j=1}^{N} \Delta K_{ij} \left[\sin(\theta_j - \theta_i) + \cos(\theta_j - \theta_i)(\Delta\theta_j - \Delta\theta_i)\right]. \end{aligned} \tag{S-25}$$

Thus, we obtain

$$\frac{\partial y'_i}{\partial \theta_j} = J_{ij}, \tag{S-26}$$

$$\frac{\partial y'_i}{\partial K_{lm}} = \begin{cases} \sin(\theta_m - \theta_l) & i = l \\ 0 & i \neq l \end{cases}, \tag{S-27}$$

where

$$J_{ij} = \begin{cases} -\sum_{s=1}^{N} K_{is} \cos(\theta_s - \theta_i) & i = j \\ K_{ij} \cos(\theta_j - \theta_i) & i \neq j \end{cases}. \tag{S-28}$$

Adding the same value to all θs of a phase-locked solution results in another phase-locked solution, and the latter cannot be distinguished from the former in terms of the circular variance V_c. Thus, we cannot determine the unique phase-locked solution for this model. However, we can set the average phase to 0, which will not ruin the generality of our argument. By assuming $\sum_{i=1}^{N} \theta_i = 0$, we can replace Eq. (S-23) with

$$y_i \equiv \omega_i + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i) - \sum_{j=1}^{N} \theta_j = 0. \tag{S-29}$$

Hence, $\partial y_i / \partial \theta_j$ can be derived as

$$\frac{\partial y_i}{\partial \theta_j} = J_{ij} - 1 \equiv \tilde{J}_{ij}. \tag{S-30}$$

The matrix $\tilde{\mathbf{J}} = (\tilde{J}_{ij})$ is full rank. Thus, we have

$$\begin{aligned}\frac{\partial \theta_i}{\partial K_{lm}} &= -\sum_{j=1}^{N} (\tilde{\mathbf{J}}^{-1})_{ij} \delta_{jl} \sin(\theta_m - \theta_l) \\ &= -(\tilde{\mathbf{J}}^{-1})_{il} \sin(\theta_m - \theta_l). \end{aligned} \tag{S-31}$$

Hence, the derivative of V_c with respect to K_{lm} is given by

$$\begin{aligned}\frac{\partial V_c}{\partial K_{lm}} &= -\frac{1}{2N} \left(C^2 + S^2\right)^{-1/2} \frac{\partial \left[\left(\sum_{i=1}^{N} \cos\theta_i\right)^2 + \left(\sum_{i=1}^{N} \sin\theta_i\right)^2\right]}{\partial K_{lm}} \\ &= \frac{-1}{N^2 r} \left(-C \sum_{i=1}^{N} \sin\theta_i \frac{\partial \theta_i}{\partial K_{lm}} + S \sum_{i=1}^{N} \cos\theta_i \frac{\partial \theta_i}{\partial K_{lm}}\right) \\ &= \frac{1}{N^2 r} \left(S \sum_{i=1}^{N} \cos\theta_i (\tilde{\mathbf{J}}^{-1})_{il} - C \sum_{i=1}^{N} \sin\theta_i (\tilde{\mathbf{J}}^{-1})_{il}\right) \sin(\theta_m - \theta_l). \end{aligned} \tag{S-32}$$

From the above analysis, we numerically obtain the IMPS by using

$$\begin{aligned}\mathrm{IMPS} &= \sum_{\langle lm \rangle} |\mathrm{SPS}_{lm}| \\ &= \sum_{\langle lm \rangle} \left| \frac{K_{lm}}{V_c} \frac{\partial V_c}{\partial K_{lm}} \right|, \end{aligned} \tag{S-33}$$

where $\langle\rangle$ is the summation over the pairs of (l, m) with $K_{lm} \neq 0$.

Analysis S3: Phase Oscillators on Path Graph

In general, nonlinearly coupled oscillator models can not be solved analytically. However, a solution can be obtained as follows for phase oscillators on a path graph. Here, we assume N oscillators are located on a path graph (Fig. 5B). Thus, on a path graph, $N-2$ vertices are of degree 2, and 2 vertices are of degree 1. We assume that $-\omega_1 = \omega_N = 1$ and $\omega_i = 0$ for $1 < i < N$. Under these assumptions, the dynamics are given by

$$\begin{aligned}\dot{\theta}_1 &= -1 + \alpha \sin(\theta_2 - \theta_1), \\ \dot{\theta}_2 &= \alpha \sin(\theta_3 - \theta_2) + \alpha \sin(\theta_1 - \theta_2), \\ &\vdots \\ \dot{\theta}_{N-1} &= \alpha \sin(\theta_N - \theta_{N-1}) + \alpha \sin(\theta_{N-2} - \theta_{N-1}), \\ \dot{\theta}_N &= 1 + \alpha \sin(\theta_{N-1} - \theta_N), \end{aligned} \tag{S-34}$$

where α is the coupling strength. Thus, all of N oscillators are spaced at regular intervals in the phase-locked state. N oscillators are located in a line every other $\Delta\theta = \sin^{-1}(1/\alpha) > 0$. We can assume $\theta_i = (i-1)\Delta\theta$ in the phase-locked state without loss of generality. The circular variance V_c is given by

$$V_c = 1 - r = 1 - \frac{1}{N} \left| \sum_{s=1}^{N} e^{\mathrm{i}(s-1)\Delta\theta} \right|. \tag{S-35}$$

$\mathbf{J}$, which is defined in Supplemental Materials Analysis S2, is expressed as

$$\mathbf{J} = \alpha\cos(\Delta\theta)\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}. \tag{S-36}$$

The IMPS of this undirectional model is obtained by using

$$\begin{aligned} \mathrm{IMPS} &= \frac{\alpha}{V_c}\sum_{\langle lm\rangle}\left|\frac{\partial V_c}{\partial K_{lm}} + \frac{\partial V_c}{\partial K_{ml}}\right| \\ &= \frac{\alpha\sin(\Delta\theta)}{V_c N^2 r}\sum_{n=1}^{N-1}|S\kappa_n^c - C\kappa_n^s - S\kappa_{n+1}^c + C\kappa_{n+1}^s|, \end{aligned} \tag{S-37}$$

where

$$\sum_{n=1}^{N}(\mathbf{J})_{in}\kappa_n^c = \cos\theta_i, \tag{S-38}$$

$$\sum_{n=1}^{N}(\mathbf{J})_{in}\kappa_n^s = \sin\theta_i. \tag{S-39}$$

Since $\mathbf{J}$ is of rank $N-1$ [1], κ_i^s and κ_i^c have uncertainty. Thus, we assume that $\kappa_1^s = \kappa_1^c = 0$ and determine κ_n^s and κ_n^c by recursively using κ_i^s $(i < n)$ to obtain

$$\begin{aligned} \kappa_n^c &= \frac{1}{\alpha\cos(\Delta\theta)}\sum_{s=1}^{n-1}(n-s)\cos[(s-1)\Delta\theta] \\ &= \frac{1}{\alpha\cos(\Delta\theta)}\mathrm{Re}\left[n\frac{z^{n-1}-1}{z-1} - \left(\frac{z^n - z}{z-1}\right)'\right] \\ \kappa_n^s &= \frac{1}{\alpha\cos(\Delta\theta)}\sum_{s=1}^{n-1}(n-s)\sin[(s-1)\Delta\theta] \\ &= \frac{1}{\alpha\cos(\Delta\theta)}\mathrm{Im}\left[n\frac{z^{n-1}-1}{z-1} - \left(\frac{z^n - z}{z-1}\right)'\right] \end{aligned} \tag{S-40}$$

where

$$z = \cos(\Delta\theta) + \mathrm{i}\sin(\Delta\theta). \tag{S-41}$$

Thus, assuming $N\Delta\theta \le 2\pi$, IMPS is derived as

$$\begin{aligned}
\text{IMPS} &= \frac{\alpha \sin(\Delta\theta)}{V_c N^2 r\alpha \cos(\Delta\theta)} \sum_{n=1}^{N-1} \left| S\text{Re}[f_n(z)] - C\text{Im}[f_n(z)] - S\text{Re}[f_{n+1}(z)] + C\text{Im}[f_{n+1}(z)] \right| \\
&= \frac{\tan(\Delta\theta)}{V_c N^2 r} \sum_{n=1}^{N-1} \left| S\text{Re}[f_n(z) - f_{n+1}(z)] - C\text{Im}[f_n(z) - f_{n+1}(z)] \right| \\
&= \frac{\tan(\Delta\theta)}{V_c N^2 r} \sum_{n=1}^{N-1} \left| \text{Im}\left(\frac{z^N - 1}{z-1} \overline{[f_n(z) - f_{n+1}(z)]} \right) \right| \\
&= \frac{\tan(\Delta\theta)}{V_c N^2 r} \left| \text{Im}\left[\frac{z^N - 1}{z-1} \overline{\left(\sum_{n=1}^{N-1} \frac{-z^n + 1}{z-1} \right)} \right] \right| \\
&= \frac{\tan(\Delta\theta)}{N^2 r(1-r)} \left| \text{Im}\left[\frac{z^N - 1}{z-1} \overline{\left(\frac{-z^N + N(z-1) + 1}{(z-1)^2} \right)} \right] \right|, \qquad \text{(S-42)}
\end{aligned}$$

where we have used

$$f_n(z) = n\frac{z^{n-1} - 1}{z-1} - \left(\frac{z^n - z}{z-1} \right)', \qquad \text{(S-43)}$$

$$\begin{aligned}
f_n(z) - f_{n+1}(z) &= n\frac{z^{n-1} - 1}{z-1} - \left(\frac{z^n - z}{z-1} \right)' - (n+1)\frac{z^n - 1}{z-1} + \left(\frac{z^{n+1} - z}{z-1} \right)' \\
&= n\frac{z^{n-1} - z^n}{z-1} - \frac{z^n - 1}{z-1} - \left(\frac{z^n - z}{z-1} - \frac{z^{n+1} - z}{z-1} \right)' \\
&= -nz^{n-1} - \frac{z^n - 1}{z-1} + nz^{n-1} \\
&= -\frac{z^n - 1}{z-1}. \qquad \text{(S-44)}
\end{aligned}$$

Figure S1: Correlation between IMPS and Average Path Length

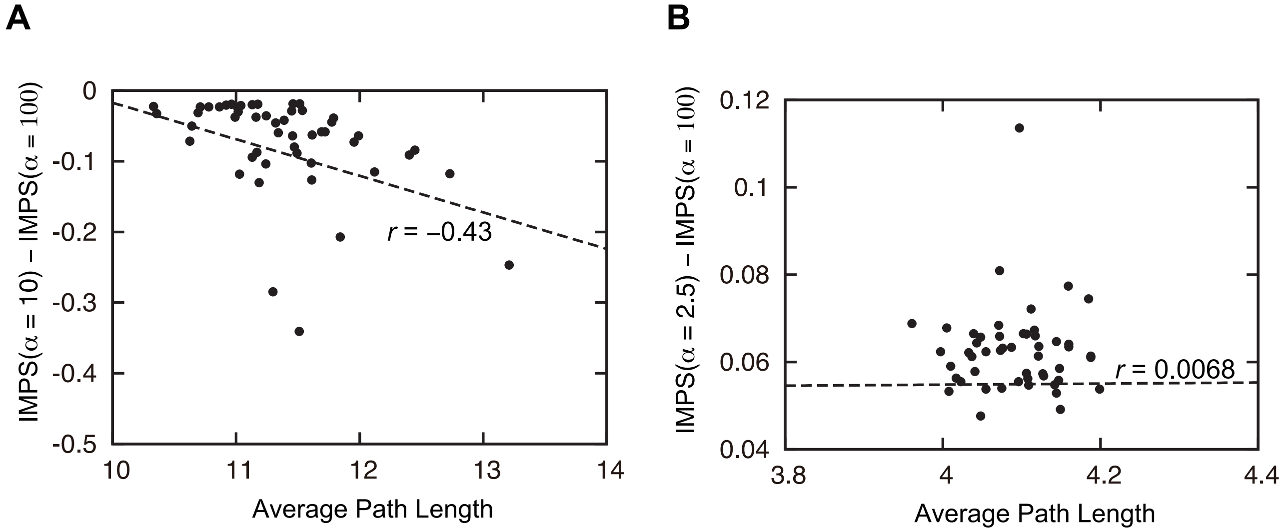

Figure S1. (A) Scatter diagram of IMPS($\alpha = 10$) − IMPS($\alpha = 100$) against average path length for 50 Watts–Strogatz networks ($N = 1000$, rewiring probability= 0.05). Correlation coefficient r is -0.43 ($p < 0.01$). (B) Scatter diagram of IMPS($\alpha = 2.5$) − IMPS($\alpha = 100$) against average path length for 50 Barabási–Albert networks ($N = 1000$). Correlation coefficient r is 0.0068 (non-significant).

References

1. R. Olfati-Saber and R. M. Murray: IEEE Trans. Automatic Control **49** (2004) 1520.

2. N. I. Fisher: *Statistical Analysis of Circular Data* (Cambridge University Press, 1995).