Invariant Multiparameter Sensitivity of Oscillator Networks

Kenzaburo Fujiwara, Takuma Tanaka, and Kiyohiko Nakamura

Department of Computational Intelligence and Systems Science, Interdisciplinary Graduate School of Science and Engineering,
Tokyo Institute of Technology, Yokohama, 226-8502, Japan
kenzaburo.fujiwara@gmail.com

Abstract. The behavior of neuronal and other biological systems is determined by their parameter values. We introduce a new metric to quantify the sensitivity of output to parameter changes. This metric is referred to as invariant multiparameter sensitivity (IMPS) because it takes on the same value for a class of equivalent systems. As a simplification of neuronal membrane, we calculate, in parallel resistor circuits, the values of IMPS and a previously studied metric of parameter sensitivity. Furthermore, we simulate phase oscillator models on complex networks and clarify the property of IMPS.

Keywords: Parameter Sensitivity, Complex Network, Phase Oscillator, Synchronization.

1 Introduction

A large number of mathematical models have been proposed in order to explain complex phenomena in brain including learning, chaotic behavior and synchronization [1, 2]. Because these models have many parameters, it would be desirable to know how changes of parameter values influence the output of a model. Information about the relationship between parameter changes and output of models is indispensable in designing models, fitting parameters and understanding the dynamics of systems [3, 4].

In this paper, we investigate parameter sensitivity, that is, the response of output to small changes of parameters. Parameter sensitivity has been intensively studied in circuit theory, particularly in resistor-capacitor networks [4–7]. In biochemical modeling, metrics of sensitivity are also used in quantifying robustness of systems [8]. In neuronal modeling and machine learning, it is important to estimate how sensitively output, such as firing rate and generalization error, changes in response to small parameter changes.

Several metrics of parameter sensitivity have been proposed in previous studies. Single parameter sensitivity (SPS) allows us to quantify the output changes in response to small change of a single parameter. Multiparameter sensitivity (MPS) is a generalization of SPS to multiple parameters [9]. MPS is defined as the square root of the sum of the square of SPSs. However, as will be shown later,

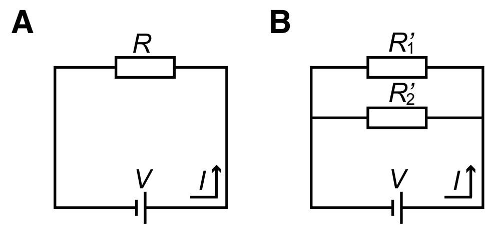

Fig. 1. Circuit A and circuit B are equivalent if $R = R'_1 R'_2/(R'_1 + R'_2)$.

MPS gives different values for such equivalent electric circuits as in Fig. 1(A) and Fig. 1(B).

We propose in this paper a new metric of sensitivity, which we call invariant multiparameter sensitivity (IMPS). This paper is organized as follows. In section 2, we introduce sensitivity metrics previously proposed and define IMPS. Then we derive basic properties of IMPS. In section 3, we examine properties of IMPS by applying it to a simple circuit. In section 4, we further investigate IMPS for nonlinearly coupled oscillators. Since it was reported that networks in brain are scale-free networks [10], in which the number of connections of each vertex obeys a power-law distribution, we examine the system of oscillators on a scale-free network. In section 5, we summarize our results and discuss potential applications.

2 Parameter Sensitivity

Dynamical systems are expressed by first-order differential equations

$$\dot{\boldsymbol{x}} = F\left(t, \boldsymbol{x}, \boldsymbol{p}\right), \tag{1}$$

where t is time, $\boldsymbol{x} = [x_1, x_2, x_3, \ldots, x_n]$ is the state variable vector and $\boldsymbol{p} = [p_1, p_2, p_3, \ldots, p_m]$ is the parameter vector.

For the output q of the system, single parameter sensitivity for parameter p_i, which we call SPS_i, is defined as

$$\mathrm{SPS}_i = \frac{p_i}{q}\frac{\partial q}{\partial p_i} = \frac{\partial \ln q}{\partial \ln p_i}. \tag{2}$$

SPS_i is the ratio of the change of output q to the change of parameter p_i. However, SPS_i does not quantify sensitivity to the change of other parameters. MPS, which is defined as

$$\mathrm{MPS}^2 = \sum_{i=1}^{m} \mathrm{SPS}_i^2, \tag{3}$$

is known as a metric to estimate sensitivity to the change of the whole parameter set of the system [9, 7]. As shown in the next section, MPS often gives different values for two equivalent models, and thereby MPS is not appropriate for comparing sensitivities between models. Thus, we introduce a new metric, invariant multiparameter sensitivity (IMPS). IMPS is defined by the sum of absolute values of SPSs as

$$\mathrm{IMPS} = \sum_{i=1}^{m} |\mathrm{SPS}_i| \,. \tag{4}$$

IMPS gives the same values for equivalent models in many cases. Assuming that $q(p_1, p_2, p_3, \ldots, p_m)$ is a homogeneous function of degree k and that SPSs in equation (4) have the same sign, we obtain

$$\begin{aligned}\mathrm{IMPS} &= \sum_{i=1}^{m} \left| \frac{p_i}{q} \frac{\partial q}{\partial p_i} \right| \\ &= \left| \sum_{i=1}^{m} \frac{p_i}{q} \frac{\partial q}{\partial p_i} \right| \\ &= |k| \,, \end{aligned} \tag{5}$$

where we used Euler's theorem

$$p_1 \left(\frac{\partial q}{\partial p_1} \right) + p_2 \left(\frac{\partial q}{\partial p_2} \right) + \cdots + p_m \left(\frac{\partial q}{\partial p_m} \right) = kq(p_1, p_2, p_3, \ldots, p_m). \tag{6}$$

Hence IMPS is constant. IMPS is invariant for all models satisfying the following conditions: (1) the outputs are expressed by homogeneous functions of parameters; and (2) SPSs take on the same sign.

3 Circuit Toy Models

In this section we examine circuit toy models. Consider that there is one resistor R in a circuit as in Fig. 1(A). We denote the electric energy consumption by W and the voltage of the voltage source by V. We assume that W is the output. MPS of this circuit equals 1. The same current-voltage relationship as the circuit shown in Fig. 1(A) can be realized by the circuits equivalent to it such as that in Fig. 1(B) if $R = R'_1 R'_2/(R'_1 + R'_2)$. MPS of the circuit in Fig. 1(B) is given by

$$\begin{aligned}\mathrm{MPS}^2 &= \sum_{i=1}^{n} \mathrm{SPS}_i^2 \\ &= \sum_{i=1}^{2} \left(\frac{R'_i}{W} \frac{\partial W}{\partial R'_i} \right)^2 \\ &= \left(\frac{R'_2}{R'_1 + R'_2} \right)^2 + \left(\frac{R'_1}{R'_1 + R'_2} \right)^2 \\ &< 1. \end{aligned} \tag{7}$$

Thus, MPS of the circuit in Fig. 1(B) is less than MPS of that in Fig. 1(A). In contrast, IMPS of the circuit in Fig. 1(B) is given by

$$\begin{aligned}\mathrm{IMPS} &= \sum_{i=1}^{n} |\mathrm{SPS}_i| \\ &= \frac{R_2'}{R_1' + R_2'} + \frac{R_1'}{R_1' + R_2'} \\ &= 1,\end{aligned} \tag{8}$$

which equals IMPS of that in Fig. 1(A). It can be easily shown that IMPS is the same for the energy consumption of the equivalent RC circuits, by which the electric properties of neuronal membrane have been modeled [1].

4 Nonlinear Model

In this section, we investigate the IMPS of the system of phase oscillators on a Barabási–Albert network as an example of neuronal networks. Barabási–Albert model is the most thoroughly studied scale-free network model [11]. We generate Barabási–Albert networks with average degree of 4. We start from 2 vertices and add a vertex with 2 edges in each step until we have N vertices.

We assume that N oscillators are connected to each other by the adjacency matrix $\mathbf{A_{BA}}$ of a Barabási–Albert network. The dynamics of oscillator i are described by

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i), \tag{9}$$

where K_{ij} is the (i, j)-element of the connection weight matrix defined by $\mathbf{K} = \alpha\mathbf{A_{BA}}$ and the natural frequency ω_i is drawn from the Gaussian distribution with unit variance. We assume that $\sum_{i=1}^{N} \omega_i = 0$ without loss of generality. Here, α is the connection strength. We use the circular variance V of the oscillators

$$V = 1 - r = 1 - \frac{1}{N}\sqrt{C^2 + S^2} \tag{10}$$

in the phase-locked state as an output, where r is the Kuramoto order parameter, $C = \sum_{i=1}^{N} \cos\theta_i$ and $S = \sum_{i=1}^{N} \sin\theta_i$.

In the phase-locked state, the right-hand side y_i' of equation (9) is 0, that is,

$$\boldsymbol{y'} = \mathbf{0}. \tag{11}$$

Here we derive the relationship between the connection weights and the phases under the condition that equation (11) is satisfied. Assuming that $\Delta\mathbf{K}$ is small, we obtain

$$
\begin{aligned}
y_i' + \Delta y_i' &= \omega_i + \sum_{j=1}^{N} (K_{ij} + \Delta K_{ij}) \sin(\theta_j + \Delta\theta_j - \theta_i - \Delta\theta_i) \\
&\approx \omega_i + \sum_{j=1}^{N} (K_{ij} + \Delta K_{ij}) \left[\sin(\theta_j - \theta_i) + \cos(\theta_j - \theta_i)(\Delta\theta_j - \Delta\theta_i)\right].
\end{aligned}
\tag{12}
$$

Subtracting y_i' from both sides yields

$$
\begin{aligned}
\Delta y_i' \approx & \sum_{j=1}^{N} K_{ij} \cos(\theta_j - \theta_i)(\Delta\theta_j - \Delta\theta_i) \\
& + \sum_{j=1}^{N} \Delta K_{ij} \left[\sin(\theta_j - \theta_i) + \cos(\theta_j - \theta_i)(\Delta\theta_j - \Delta\theta_i)\right].
\end{aligned}
\tag{13}
$$

Thus we obtain

$$
\frac{\partial y_i'}{\partial \theta_j} \equiv J_{ij}', \tag{14}
$$

$$
\frac{\partial y_i'}{\partial K_{lm}} = \begin{cases} \sin(\theta_m - \theta_l) & i = l \\ 0 & i \neq l \end{cases}, \tag{15}
$$

where

$$
J_{ij}' = \begin{cases} -\sum_{s=1}^{N} K_{is} \cos(\theta_s - \theta_i) & i = j \\ K_{ij} \cos(\theta_j - \theta_i) & i \neq j \end{cases}. \tag{16}
$$

J' is of $N-1$ rank, because Laplacian matrices of connected graphs are of $N-1$ rank [12]. Adding the same value to all θ_i's of a phase-locked solution results in another phase-locked solution, and the latter cannot be distinguished from the former in terms of V. Thus we cannot determine the unique phase-locked solution for this model. However, we can set the average phase to 0, which will not ruin the generality of our argument, because we are interested only in circular variance V of the oscillators. Assuming $\sum_{i=1}^{N} \theta_i = 0$, we can replace equation (11) with

$$
y_i \equiv \omega_i + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i) + \sum_{j=1}^{N} \theta_j = 0. \tag{17}
$$

Hence $\partial y_i / \partial \theta_j$ can be derived as

$$
\frac{\partial y_i}{\partial \theta_j} = J_{ij}' + 1 \equiv J_{ij}. \tag{18}
$$

J is full rank. Thus we have

$$\frac{\partial \theta_i}{\partial K_{lm}} = -\sum_{j=1}^{N} (\mathbf{J}^{-1})_{ij} \delta_{jl} \sin(\theta_m - \theta_l)$$
$$= -(\mathbf{J}^{-1})_{il} \sin(\theta_m - \theta_l). \tag{19}$$

Hence the derivative of V with respect to K_{lm} is given by

$$\frac{\partial V}{\partial K_{lm}} = -\frac{1}{2N} \left(C^2 + S^2\right)^{-1/2} \frac{\partial \left[\left(\sum_{i=1}^{N} \cos \theta_i \right)^2 + \left(\sum_{i=1}^{N} \sin \theta_i \right)^2 \right]}{\partial K_{lm}}$$
$$= \frac{1}{N^2 r} \left(S \sum_{i=1}^{N} \cos \theta_i (\mathbf{J}^{-1})_{il} - C \sum_{i=1}^{N} \sin \theta_i (\mathbf{J}^{-1})_{il} \right) \sin(\theta_m - \theta_l). \tag{20}$$

From the above analysis, we numerically obtain IMPS as

$$\text{IMPS} = \sum_{\langle lm \rangle} |\text{SPS}_{lm}| = \sum_{\langle lm \rangle} \left| \frac{K_{lm}}{V} \frac{\partial V}{\partial K_{lm}} \right|, \tag{21}$$

where $\langle\rangle$ is the summation over the connected oscillator pairs. In the initial state, all phases are uniformly distributed. When α is sufficiently large, the oscillators are phase locked as shown in Fig. 2. We calculate the IMPS for various values of α under phase-locked conditions.

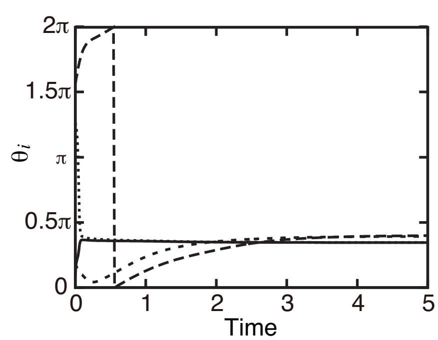

Fig. 2. Synchronization of 1000 oscillators on a Barabási–Albert network with average degree of 4. Phases of 4 out of 1000 oscillators are shown. The connection strength α is set to 2.

The IMPS of this model is shown in Fig. 3. IMPS gives similar values for system size $N = 1000$ (Fig. 3A) and 10000 (Fig. 3B), whereas MPS exhibits system-size dependency (Fig. 3C). Unlike the toy circuit model in section 3,

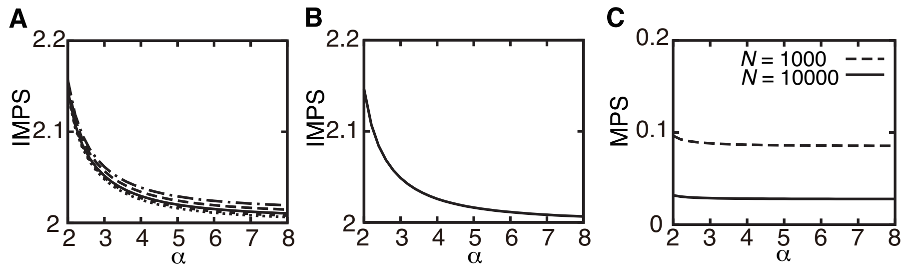

Fig. 3. IMPS of oscillator networks with $N = 1000$ (A) and $N = 10000$ (B). Panel A shows the results of networks generated by 5 different random seeds. Panel C shows the MPS for $N = 1000$ (dashed line) and the MPS for $N = 10000$ (solid line).

IMPS for this system of phase oscillators does not take on a constant value because of the nonlinearity of this system. If α is sufficiently large, the nonlinearity of sine coupling is ignorable. Therefore, as α increases, IMPS of this oscillator system converges to 2 because the circular variance V of this model converges to a homogeneous function of degree -2.

5 Discussion

In this paper, we have reviewed the previously proposed metrics, SPS and MPS, which quantify parameter sensitivity. We have formulated an improved metric, IMPS. IMPS gives the same value for equivalent models in many cases. This gives IMPS a significant advantage over MPS, which gives different values for equivalent systems. In the analysis of the simple circuits, IMPS has given the same value for equivalent parallel circuits. Then we have applied IMPS to nonlinear complex systems. As a first step for applying IMPS to neuronal systems, we have used the phase oscillator model and the Barabási–Albert model because those two models are widely used in the previous research [11, 13, 14].

Formerly, invariance of IMPS was reported only for RC network circuits [4–7]. In this paper, we have shown its invariance in a wider setting than previous studies. In the system of phase oscillators, IMPS is not always invariant because of the nonlinearity. Our results suggest that IMPS is a metric reflecting both structure and dynamics of the systems. Thus IMPS would allow us to estimate the dynamics and excitability of individual neurons and synaptic connectivity between neurons.

In a future work, it should be examined how structure and nonlinearity of systems are reflected in the value of IMPS. In particular, we will apply IMPS to the system of neurons on a Watts–Strogatz small-world network [15]. Furthermore, the relation between IMPS and previously proposed network metrics, such as cluster coefficient and average path length [15, 16], should be investigated. The application of IMPS to the real neuronal networks is also of interest.

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